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The local structure of anomaly inflow

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Abstract: Anomaly cancellation for M-theory five-branes requires the introduction of a "bump-form" which smoothes out the five-brane source. We discuss the physical origin of this bump-form in the simpler case of axion strings in $3+1$ dimensions and construct it in terms of the radial profile of the fermion zero modes. Our treatment allows for a clearer understanding of the role played by covariant rather than consistent anomalies when anomalies are canceled by inflow from the bulk. We briefly discuss the generalization of these results to five-brane anomalies in M-theory.

Keywords: M-Theory, Anomalies in Field and String Theories.

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Contents

1. Introduction

Anomalies are known to be an important tool for getting non-perturbative information about quantum field theories. The requirement of cancellation of anomalies in local gauge symmetries gives important constraints on the structure of the theory, and anomalies in global symmetries often give important non-perturbative information about the theory.

In the case of M-theory where we lack a formulation that is both complete and practical for computation, the study of anomalies is especially important as a means of obtaining exact results. An analysis of diffeomorphism anomalies in the presence of a M5-brane ([1]–[5]), shows that the "standard" mechanism of cancellation of anomalies by inflow from the bulk does not work in the expected way. Namely, there remains a *normal bundle* anomaly, coming from the part of the diffeomorphism group which acts on the fermion zero modes as SO(5) gauge transformations by rotating the normal bundle at each point on the brane. There were a number of attempts to treat this problem. In [2] it was shown that if the theory is compactified on a circle (i.e. going to type-IIA string theory) then the anomaly can be canceled by a local counterterm on the brane. In [4] a local counterterm was suggested for the eleven-dimensional case (with the result of [2] restored upon compactification). However this work is based on the fact that the world-volume of five-brane is closed, being the boundary of some seven-dimensional manifold. This cannot be considered to be a totally satisfactory answer, as branes can be taken to have boundaries and as there is no physical meaning so far to this seven-dimensional manifold.

The work [3] was based on the idea that anomaly cancellation in the presence of the Chern-Simons term in eleven-dimensional supergravity requires smoothing out the M5-brane source. By smoothing the brane, modifying the Chern-Simons term in the bulk action, it was shown that the anomaly of the normal bundle indeed cancels. The relation of this mechanism to the cancellation in type-IIA theory was discussed in [5]. The price paid for this anomaly canceling mechanism was the presence in the action of the *bump function*: an arbitrary function of the distance from the brane with definite boundary values. So the question remained: what is the physical origin of the bump function $\rho(r)$ which enters the action and is required for anomaly cancellation?

In what follows we will consider the role of such a bump function in a much simpler model where its physical origin can be analyzed. It is connected to the relation between consistent and covariant anomalies and can be related to the radial profile of fermion zero modes on the brane. Following this analysis we make some remarks on the extension of these results to the M5-brane. It should be noted that cancellation of anomalies can be phrased in global topological terms [6] for which the present considerations are irrelevant. Rather, we are interested here in understanding the detailed local structure of anomaly cancellation from a physical point of view. Such considerations might be relevant in applications of anomaly inflow to condensed matter systems [7, 8, 9] and the description of chiral fermions in lattice gauge theory [10].

2. Axion electrodynamics

The example we are going to deal with is *Axion Electrodynamics*. As we will see, when analyzed in detail, it possesses some features similar to those of the five-brane case. The bulk theory is anomalous in the presence of a topological defect (a $1+1$-dimensional string in this case), as is the theory on the defect. Inflow from the bulk cancels this anomaly, but a complete understanding of the cancellation requires a detailed treatment of the string source, which takes into account its profile. We begin with a quick review of material from [11, 12] and then go on to discuss the role of the bump-form and its physical origin.

Consider a theory in 3+1 dimensions, which describes Dirac fermions interacting with a U(1) gauge field and a complex scalar field (which is neutral under the U(1) gauge symmetry). The lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}i\not{D}\psi + |\partial_\mu\Phi|^2 + g\bar{\psi}(\Phi_1 + i\gamma^5\Phi_2)\psi - V(\Phi)\,. \tag{2.1}$$

The complex scalar field $\Phi = \Phi_1 + i\Phi_2$ has a potential $V(\Phi)$ which we take to have the form

$$V(\Phi) = \lambda(|\Phi|^2 - v^2)^2\,. \tag{2.2}$$

The equations of motion admit a global vortex solution of the form:

$$\Phi(x^\mu) = f(x_\perp)e^{i\theta(x^\mu)}\,. \tag{2.3}$$

Here $x_\perp = \sqrt{x^2+y^2}$ is the radial coordinate in the (x,y)-plane, and the phase $\theta(x^\mu)$ gives the classical value of the *axion* field. The profile function $f(x_\perp)$ is assumed to be zero at the origin and equal to the v far from it. For a vortex with winding number one the axion field is $\theta(x^\mu) = \varphi$, where φ is a polar angle in the (x,y) plane. This configuration (known as an *axion string*) describes a topological defect with codimension 2 and topological charge one.[1]

2.1 Fermion zero modes and anomaly inflow

The index theorem guarantees that the Dirac equation in the background (2.3) has a *zero-mode* solution, i.e.

$$\left(i\partial\!\!\!/ + f(x_\perp)e^{i\gamma^5\varphi}\right)\psi_{\text{z.m.}} = 0 \tag{2.4}$$

(we've absorbed the Yukawa coupling constant g into the function f throughout the rest of the paper). The zero mode solution is [11]:

$$\psi_{\text{z.m.}} = u_\alpha e^{-ip(t+z)}\mathcal{F}(x_\perp)\,, \tag{2.5}$$

where we have introduced the function $\mathcal{F}$ which is the radial profile of the fermion zero mode:

$$\mathcal{F}(r) = \mathcal{C}e^{-\int_0^r f(\sigma)d\sigma} \tag{2.6}$$

with $\mathcal{C}$ a normalization constant. The spinor u_α has the form $u = (1-i\gamma^1)\eta$. Here spinor η obeys $\gamma^{int}\eta = -\eta$ and $\gamma^5\eta = -\eta$ with $\gamma^{int} = \gamma^0\gamma^3$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. In the conventions of [13] it has the explicit form

$$u_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix}. \tag{2.7}$$

Note that this solution decays exponentially in the (x,y)-plane, being quasi-two-dimensional and describes a fermion of negative (two-dimensional) chirality, propagating in the $-z$ direction. In order to obtain canonically normalized two-dimensional fermions we impose the normalization condition

$$\int d^2x_\perp\, dz\, \psi_p^*\psi_{p'} = 2\pi\delta(p-p') \tag{2.8}$$

[1]An axion string with topological charge n would have $\theta(x^\mu) = n\varphi$.

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which in particular determines the normalization constant $\mathcal{C}$ from the condition:

$$\int d^2x_\perp \, \mathcal{F}^2(x_\perp) = 1 \,. \tag{2.9}$$

The theory of fermions of one chirality in two dimensions is anomalous when coupled to an electromagnetic field[2]

$$\partial_a j^a = -\frac{e}{8\pi}\epsilon^{ab}F_{ab} \tag{2.10}$$

and thus violates charge conservation. In our case, however, this theory is naturally embedded into a non-anomalous $3+1$-dimensional theory and charge conservation should be ensured by anomaly inflow from the bulk.

Using the ideas of [14] one can compute the current far from the string in the presence of a background electromagnetic field by computing a one-loop Feynman diagram with one insertion of the electromagnetic field and one insertion of the scalar field. One finds ([11, 12]):

$$\langle j^\mu \rangle = \frac{e}{8\pi^2}\epsilon^{\mu\nu\lambda\sigma}\partial_\nu \theta F_{\lambda\sigma} = \frac{e}{4\pi^2}\partial_\nu(\theta \tilde{F}^{\mu\nu}) \,. \tag{2.11}$$

In the presence of a constant electric background field pointing along the axis of the axion string this gives a current flowing radially inward and thus bringing electric charge to the string. This is the anomaly inflow we are looking for. The divergence of the current on the string can be computed using the fact that

$$[\partial_x, \, \partial_y]\theta = 2\pi\delta^{(2)}(x_\perp) \tag{2.12}$$

to give

$$\partial_\mu \langle j^\mu \rangle = \frac{e}{4\pi}\epsilon^{ab}F_{ab}\delta^2(x_\perp) \tag{2.13}$$

twice the expression (2.10)!

This discrepancy was explained in [12] as arising from the difference between the covariant and consistent anomaly which for an abelian gauge theory in $1+1$ dimensions is simply a factor of two [15]. The anomaly (2.10) is the consistent anomaly in $1+1$ dimensions, following from variation of a $1+1$-dimensional action. The current (2.11) can be obtained from the action:

$$S_{\text{eff}} = -\frac{e^2}{16\pi^2}\int_{M_4} \epsilon^{\mu\nu\lambda\sigma}\partial_\mu \theta A_\nu F_{\lambda\sigma} \tag{2.14}$$

which up to an integration by parts is the usual coupling of the axion to gauge fields which results from integrating out the fermion fields. This action is covariant as is the resulting current.

It was proposed in [12] that there should be an additional contribution to the current on the string which converts the covariant current flowing in from infinity into a covariant current on the axion string. It was also noted there that such a term

[2] In this paper Latin indices (a, b) run over subset $\{0, 3\}$ and Greek indices run $0, \ldots, 3$.

could be derived formally by treating the action (2.14) as valid everywhere, not just far from the string. Variation of the action then leads to an addition contribution to the current which is delta-function localized on the string and is of precisely the right form to convert the consistent current on the string into the covariant current.

2.2 Anomaly cancellation revisited

From the analysis in the last section we see several reasons why we would like to better understand the physics of anomaly inflow. First, the naive analysis shows that divergence of the current flowing in to the string does not match the divergence of the current on the string computed in the 1+1-dimensional theory. The modification suggested in [12] extrapolates the formula (2.14) to the whole bulk. It is obvious, however, from the way this formula has been derived that it cannot be valid in the region near the string. Second, usage of equations similar to (2.12) is hard to justify rigorously, as neither θ nor $d\theta$ are well defined at the origin and its second derivative is ambiguous. The main reason, however, and the original motivation was the hope that we will be able to understand better the procedure used in [3] to smooth out the five-brane source. In the current case we are in much better shape though, as both the form of the original lagrangian and the fundamental degrees of freedom are known. So, we may be able to shed some light on the question: "What is the ρ-function?"

To summarize: we are hoping to derive an effective action in the bulk with the following properties:

(1) It is well defined in the whole M_4 — four-dimensional bulk and not only far from the string.

(2) The divergence of the current following from this action matches the divergence of the current from the anomaly of the string zero modes.

(3) The string source is described as a smooth solution and the delta-function equations (2.12) are modified — that is smoothed out.

Following [3] we will first postulate an interaction term in the effective action given by

$$S_{\text{eff}} = -\frac{e^2}{16\pi^2}\int_{M_4} \epsilon^{\mu\nu\lambda\sigma}(1+\rho)\partial_\mu\theta A_\nu F_{\lambda\sigma}\,. \tag{2.15}$$

Here we've introduced $\rho(r)$, a monotonic function (zero-form) of the transverse radius r with the properties:

$$\rho(0) = -1\,; \qquad \rho(r\to\infty) = 0\,. \tag{2.16}$$

This imply in particular that:

$$\int_0^\infty dr\,\rho'(r) = \rho(\infty) - \rho(0) = 1\,. \tag{2.17}$$

The action (2.15) is well defined everywhere in M_4, as the form $d\varphi$ (in the future we do distinguish between $d\theta$ and $d\varphi$, considering the axion field to be in its background value) is multiplied by $(1+\rho)$, which is zero on the string (at $r=0$).

A function with the properties (2.16)–(2.17) can be used to define a smooth generalization of the delta-function describing the embedding of the string worldsheet into spacetime: $\Sigma_2 \hookrightarrow M_4$, similar to the Poincare dual of the submanifold (see e.g. [16]), i.e. the form (rather cohomology class) $\Omega_2^{(P)}$ such that for any 2-form ω_2 on M_4:

$$\int_{M_4} \Omega_2^{(P)} \wedge \omega_2 = \int_{\Sigma_2} \omega_2 \,. \tag{2.18}$$

In terms of the *bump-form* $d\rho = \rho'(r)dr$ (where ρ has properties as above), a representative of the Poincare dual class can be written as:

$$\Omega_2^{(P)} = \frac{1}{2\pi} d\rho \wedge d\varphi \,. \tag{2.19}$$

This object does have all the properties we would like for a smoothed δ-function. From this point on whenever we write $\delta^2(x_\perp)$ we will actually imply its smoothed version (2.19) and will use the δ-function symbol mostly for convenience.

Varying eq. (2.15) one can get the expression for the current:

$$j^\mu = \frac{e}{8\pi^2}\epsilon^{\mu\nu\lambda\sigma}(1+\rho)\partial_\nu\theta F_{\lambda\sigma} + \frac{e}{8\pi^2}\epsilon^{\mu\nu\lambda\sigma}(\partial_\lambda\rho)\,(\partial_\nu\theta)\,A_\sigma \,. \tag{2.20}$$

In this case the integration by parts does not produce a surface term, as it is multiplied by $(1+\rho)$ which is zero at the surface of the string. The second term in the eq. (2.20) can be rewritten as (note that $\partial_i\rho \neq 0$ only for $i=1,2$):

$$\Delta j^\mu = -\frac{e}{8\pi^2}\begin{cases} 0\,, & \mu \in \{1,2\}\,, \\ \epsilon^{\mu\nu}\,A_\nu\, d\rho\wedge d\varphi\,, & \mu\in\{0,3\} \end{cases} \tag{2.21}$$

(here we've defined $\epsilon^{\mu\nu} = \epsilon^{ab}$ $\mu,\nu = \{0,3\}$, 0 otherwise). This is almost the same as the additional contribution to the current (2.11) found in [12]. There it was shown that to explain discrepancy of the factor of two between anomalies in the bulk (given by (2.13)) and on the string (eq. (2.10)) an addition to the current in the form (2.21) was needed. The full current in our case will be:

$$j^\mu = \frac{e}{8\pi^2}\epsilon^{\mu\nu\lambda\sigma}(1+\rho)\partial_\nu\theta F_{\lambda\sigma} - \frac{e}{4\pi}\epsilon^{\mu\nu}A_\nu\delta^2(x_\perp) \tag{2.22}$$

(the only difference with [12] being the factor $(1+\rho)$ in the first term and the fact that the delta function in the second term is now the smoothed out version (2.19)). The divergence of this current integrated over M_4 is

$$\int_{M_4}\partial_\mu j^\mu = \int_{M_4}\frac{e}{8\pi^2}\epsilon^{\mu\nu\lambda\sigma}\partial_\mu\rho\partial_\nu\theta\partial_\lambda A_\sigma = \int_{\Sigma_2}\frac{e}{4\pi}\epsilon^{ab}\partial_a A_b = \int_{\Sigma_2}\frac{e}{4\pi}F_{03} \tag{2.23}$$

which correctly cancels the anomaly (2.10).

2.3 Computation of the determinant of the Dirac operator

We now want to derive (2.15) from first principles. We need in principle to implement the usual procedure of QFT: start from the partition function of the full theory (fermions coupled to all the fields which we call $\mathcal{A}$ for the moment):

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\mathcal{A}\, e^{iS[\psi,\mathcal{A}]}$$

and integrate out the fermions to get:

$$Z = \int \mathcal{D}\mathcal{A}\, \det(i\not{\partial} + \mathcal{A}) = \int \mathcal{D}\mathcal{A}\, e^{iS_{\mathrm{eff}}[\mathcal{A}]} . \tag{2.24}$$

The determinant of the Dirac operator (2.24) (if computed exactly) will then be a well-defined gauge invariant object. The effective action (2.14), however, was obtained using only expressions for the wave functions far from the string, i.e. we took into account only part of the spectrum and so it is not surprising that we find a lack of gauge invariance. Moreover, the effective action of the form (2.14) does not contain any information about the structure of the theory near the string, treating it as a singular source. Solutions of the Dirac equation, used in the computations in the previous section which led to (2.14) were just plain waves, i.e. the solutions of the eq. (2.4) in the topologically trivial background $f(x) = v$. To obtain the action in a form (2.15) we need to bring some information about the behavior of the theory near the string.

Said another way, one could try to construct the effective action by doing a Taylor series expansion of $f(x_\perp)$ and including the radial variations in a perturbative calculation of the current. One quickly sees however that this does not change the current at any finite order in perturbation theory, essentially because the perturbative current is independent of the magnitude of f. Instead we must go beyond perturbation theory about a plane wave basis of fermion states by including the explicit form of the zero mode wave function in the calculation. That is we would like to compute the effective action non-perturbatively in $f(x_\perp)$ while still in the lowest order in A_μ.

In principle, to obtain the effective action in the bulk we would need to compute:

$$\det(i\not{\partial} + \not{\mathcal{A}}) = \prod \lambda_n , \tag{2.25}$$

where λ_n are eigenvalues of the Dirac operator in the external field $\mathcal{A}$:

$$(i\not{\partial} + \not{\mathcal{A}})\psi_n = \lambda_n \psi_n . \tag{2.26}$$

We are not really interested directly in the determinant. Rather, we are interested in the current which arises from varying the determinant with respect to the gauge field and the decomposition of this current into a part arising from the zero modes, and the rest which we wish to describe in terms of an effective action. We propose to study this decomposition in the following way.

The physical picture we have so far is that the current flowing radially inward from infinity toward the string (with divergence given by eq. (2.13)) transforms into the current along the string (with the divergence which should be twice the expression (2.10)). We know, that the gauge variation of the full $\det(i\partial\!\!\!/ + A\!\!\!/)$ is proportional to the divergence of the current $\partial_\mu j^\mu$. As the full current is conserved, it means that the contribution to the divergence of the four-dimensional current coming from the zero modes only (we will call this part of the current $j^\mu_{\rm z.m.}$) is opposite to that of coming from the effective action (2.15). Thus by computing the divergence of $j^\mu_{\rm z.m.}$ we will be able to match it against the gauge variation of (2.15) and extract the expression for $\rho(r)$.

Computation of the zero mode current $j_{\rm z.m.}$ is a much easier task by itself. Zero mode solutions (2.5) are the only contributions to it. The reason for this is very simple — any other solution of the Dirac equation (2.4) will be massive from the $1+1$-dimensional point of view and will not contribute to the axial anomaly.

Consider the standard fermion operator

$$\hat\psi(x) = \sum_n \psi_n(x)\hat a_n\,. \tag{2.27}$$

Here ψ_n's are the (exact) eigenfunctions of the Dirac operator in the background field (2.3) and $\hat a_n$ are creation/annihilation operators of the particles (we will do all this in much more detail in the next section). Now, split the r.h.s. of the expression (2.27) into two terms: $\hat\psi_{\rm z.m.}$ which is constructed solely in terms of solutions (2.5) and $\hat\psi_{\rm non\,z.m.}$ — all the other solutions. The Green's function $G(x,x') = \langle\hat\psi\hat\psi^+\rangle$ naturally splits then into $G_{\rm z.m.}(x,x') + G_{\rm non\,z.m.}(x,x')$, where:

$$G_{\rm z.m.}(x,x') = \left\langle \hat\psi_{\rm z.m.}(x)\hat\psi^+_{\rm z.m.}(x')\right\rangle . \tag{2.28}$$

The zero mode contribution to the full four-dimensional current will be:

$$\langle j^\mu(x)\rangle_{\rm z.m.} = (-ie)(-1)\int d^4y\,{\rm Tr}\Big(\gamma^\mu G_{\rm z.m.}(x,y)\gamma^\nu G_{\rm z.m.}(y,x)\Big)A_\nu(y)\,. \tag{2.29}$$

2.3.1 Computation of the current

Now we will implement in detail the procedure described in the previous section. First, we construct the Green's function $G_{\rm z.m.}$ of the zero modes in the background scalar field Φ (using the function (2.5) only). Then using $G_{\rm z.m.}$ we compute the expectation value of the current to lowest order in the electromagnetic field (2.29). This is just the standard Feynman loop diagram shown on figure 1, where the full Green's functions $G(x,x')$ are substituted with zero mode Green's functions $G_{\rm z.m.}$. We will read off the gauge variation of the effective action in the bulk from the divergence of this current.

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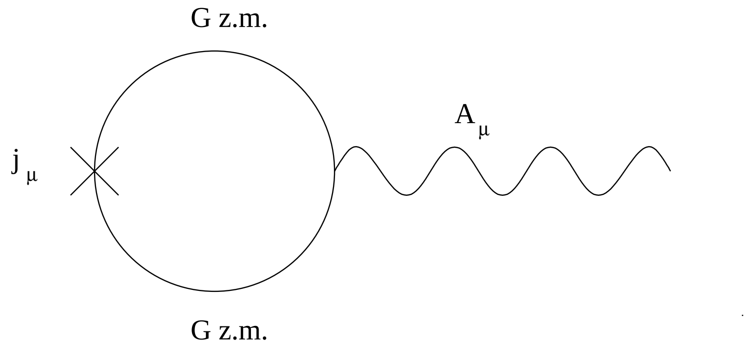

Figure 1: Contribution to the current from zero modes.

To construct Green's function we write the ψ-operators for the zero mode solution (2.5) first. Note, that in the eq. (2.5) $p > 0$ corresponds to the particle and $p < 0$ – to the antiparticle, both propagating in the same direction. Thus we write:

$$\hat{\psi}_\alpha(t, z|r) = \left[\int_0^\infty \frac{dp}{2\pi}\, u_\alpha\, \hat{a}_p + \int_{-\infty}^0 \frac{dp}{2\pi}\, u_\alpha\, \hat{b}^+_{-p}\right] e^{-ip(t+z)} \mathcal{F}(r)\,. \tag{2.30}$$

Here $\hat{a}_p$ — annihilates particle with momentum p along the string, $\hat{b}^+_p$ — creates antiparticle with the same momentum. They obey standard anticommutation relations. The Green's function is defined by the usual expression:

$$G(x,\, x') = \begin{cases} \langle \psi(x)\, \bar{\psi}(x')\rangle\,, & t > t'\,, \\ -\langle \bar{\psi}(x')\, \psi(x)\rangle\,, & t < t'\,. \end{cases} \tag{2.31}$$

From (2.30):

$$\langle \psi_\alpha(x^+|r)\, \bar{\psi}_\beta(x'^+|r')\rangle = \int_0^\infty \frac{dp}{2\pi}\frac{dp'}{2\pi} u_\alpha \bar{u}_\beta e^{-ipx^+} e^{ip'x'^+} \langle a_p a^+_{p'}\rangle \mathcal{F}(r)\mathcal{F}(r') \tag{2.32}$$

(we've introduced standard "light-cone" coordinates: $x^\pm = t \pm z$) and analogously for $\langle \bar{\psi}_\beta(x')\, \psi_\alpha(x)\rangle$. As a results:

$$G_{\rm z.m.}(x^+,\, x'^+|r, r') = \frac{1}{2\pi i}\, ||U_{\alpha\beta}||\, \mathcal{F}(r)\mathcal{F}(r') \times \frac{1}{x^+ - x'^+}\,. \tag{2.33}$$

We see that Green's function factorizes into (t, z) and (x, y) parts, and that in its (t, z) part it is a standard two-dimensional Green's function of one chiral fermion. In the above we have defined $U_{\alpha\beta} = u_\alpha \otimes \bar{u}_\beta$.

Now we are in a position to compute the expectation value of the electromagnetic current (2.29). We do the gamma-matrix algebra first:

$$\mathrm{Tr}\,(\gamma^\mu U \gamma^\nu U) = \begin{cases} 1\,, & \mu = \nu = 0\,, \quad \text{or} \quad \mu = \nu = 3\,, \\ -1\,, & \mu = 0\,,\ \nu = 3\,, \quad \text{or} \quad \mu = 3\,,\ \nu = 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.34}$$

Then it immediately follows from eqs. (2.29) and (2.34) that

$$\begin{aligned}\langle j^{1,2}(x)\rangle_{\text{z.m.}} &= 0\,,\\ \langle j^{0}(x)\rangle_{\text{z.m.}} &= -\langle j^{3}(x)\rangle_{\text{z.m.}}\,.\end{aligned} \tag{2.35}$$

This is in accordance with the fact, that for the fermions of negative chirality in two dimensions only the j_+ component of the current is non zero. Equation (2.35) implies that anomaly defined by $\mathcal{a} = \partial_\mu j^\mu$ is:

$$\mathcal{a} = (\partial_0 - \partial_3)\langle j^0(x)\rangle_{\text{z.m.}} = 2\partial_- j_+(x)\,. \tag{2.36}$$

Using the explicit form of the Green's function (2.33) and taking A_μ to be a function of (t, z) it is straightforward to compute the anomaly:

$$\mathcal{a} = \frac{e}{\pi}\partial_+ A_-(x)\mathcal{F}^2(r)\,. \tag{2.37}$$

This result is a product of two factors: $\mathcal{a}_{1+1}$ — coming from (t, z) part of the computation and the contributions from transversal directions — $\mathcal{F}(x_\perp)$. For $\mathcal{a}_{1+1}$ we have:

$$\mathcal{a}_{1+1} = \frac{e}{\pi}\partial_+ A_-(x)\,. \tag{2.38}$$

This coincides with the usual result for two-dimensional anomaly (compare [12, 17]). The standard result is usually expressed in a more symmetric way:

$$\tilde{\mathcal{a}}_{1+1} = \frac{e}{2\pi}(\partial_+ A_- - \partial_- A_+) = -\frac{e}{8\pi}\epsilon^{ab}F_{ab}\,. \tag{2.39}$$

It differs from eq. (2.38), which is just the well-known fact that one is free to add any local counterterm ($\int d^2x A^2(x)$ in this case) to the action and thus modify the anomaly from the form (2.38) to the form (2.39).

As a consistency check, the total charge non conservation (as viewed from infinity) should coincide with that of $1+1$-dimensional anomaly. This will be the integral of (2.37) over the transversal space:

$$\int d^2x_\perp \mathcal{a} = \mathcal{a}_{1+1} = -\frac{e}{8\pi}\epsilon^{ab}F_{ab} \tag{2.40}$$

(we will not distinguish between the form (2.39) and (2.38)). Thus, variation of S_{eff} in the bulk under the gauge transformation $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\Lambda$ will be:

$$\delta_\Lambda S_{\text{eff}} = -\int d^4x\,(\partial_\mu j^\mu_{\text{z.m.}})\,\Lambda = \int d^4x\,\mathcal{a}\,\Lambda\,. \tag{2.41}$$

Substituting the result (2.37) into (2.41) and using polar coordinates in the transversal direction we get:

$$\delta_\Lambda S_{\text{eff}} = -\frac{e}{8\pi}\int dr\,d\varphi\,\underbrace{r\mathcal{F}^2(r)}\int d^2x^a\,\epsilon^{ab}F_{ab}\Lambda(x)\,. \tag{2.42}$$

To extract the value of the $\rho(r)$ function from eq. (2.42), note that it follows from eq. (2.15) that

$$\delta_\Lambda S_{\rm eff} = -\frac{e}{8\pi}\int_{M_4} dr d\varphi d^2x^{int}\rho'(r)\epsilon^{ab}F_{ab}\Lambda(x)\,, \tag{2.43}$$

where we've used the expression (2.19) for the delta-function embedding the string's world-sheet into the bulk space. From (2.43), we see that the underbraced expression in (2.42) plays the role of $\frac{1}{2\pi}\rho'(r)$. Indeed, $r\mathcal{F}^2(r) = 0$ as $r \to 0$ and to ∞. The normalization condition (2.9) is enough to satisfy the property (2.17). So, the natural claim would be that:

$$\rho(r) = \frac{1}{2\pi}\int_0^r d\sigma\sigma\mathcal{F}^2(\sigma)\,. \tag{2.44}$$

We see, that it is possible to write an expression for the effective action in the whole bulk, taking into account the structure of the zero modes of the theory. Naively, one might have expected the axion string profile $f(r)$ to be a candidate for the ρ-function. It is the function $\mathcal{F}(r)$ however, which describes the zero mode behavior in the bulk, so eq. (2.44) tells us that the bump-form is related to the profile of the zero modes $\mathcal{F}(r)$, rather than to the profile of the scalar field.

3. Discussion

As would be expected physically, the analysis here shows that the anomalous divergence of the current on the axion string is spread out over a region whose size is determined by the radial wave function of the fermion zero modes. This requires a corresponding dependence on the zero mode profile in the non-zero mode contribution to the action which is reflected in the bump function and the modification to the action given in (2.15).

In the mechanism for M-theory five-brane anomaly cancellation of [3] a bump function also appears in the modification of the Bianchi identity for the four-form field strength,

$$\frac{dG_4}{2\pi} = d\rho\wedge\frac{e_4}{2}\,, \tag{3.1}$$

where $e_4/2$ is the global angular form. A naive extension of the analysis of this paper suggests that this bump function should be expressed in terms of the zero mode radial wave function as

$$d\rho(r) = r^4\mathcal{F}^4(r)dr\,. \tag{3.2}$$

The radial profile $\mathcal{F}(r)$ has power law fall-off away from the five-brane as compared to the exponential fall-off for the axion string. For example, the gravitino variation has the form

$$\begin{aligned}\delta\Psi^\mu &\propto \left(1+\frac{1}{r^3}\right)^{-3/2}\cdot\frac{1}{r^3}\,,\\ \delta\Psi^m &\propto \left(1+\frac{1}{r^3}\right)^{-7/6}\cdot\frac{1}{r^3}\,.\end{aligned} \tag{3.3}$$

In addition, cancellation of the M-theory five-brane anomaly requires non-local modification of the Chern-Simons couplings $\int C_3 \wedge G_4 \wedge G_4$. Our analysis suggests that these modification should not be viewed as corrections to the fundamental action of M-theory, but rather arise from the splitting of the action into a zero-mode action on the five-brane and an effective bulk action away from the five-brane. A microscopic derivation of these modifications would be desirable, but the axion string we have discussed is not a good model for understanding these modifications. The axion string in four dimensions does have an uncancelled normal bundle anomaly of $p_1(N)/6$, but this is trivially canceled by a local counterterm in the world-sheet effective action [2, 5].

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